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Characterizations of Classical and Quantum Logics

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In classical logic (Boolean algebras) probability systems involving correlations are fully characterized by the system of generalized Bell inequalities. On the other hand, probability systems with pairwise correlations on orthomodular lattices (OML) representing quantum logics are so general that the only inequalities that hold universally are the trivial inequalities $0 \leq p_i \leq 1$, $0 \leq p_{ij} \leq \min\{p_i, p_j\}$. In this paper it is shown that every correlation sequence $p = (p_1, \ldots, p_n, \ldots, p_{ij}, \ldots)$ satisfying the above inequalities can be represented by a probability measure on an orthomodular lattice L admitting a full set of $\{0, 1\}$-valued probability measures with the additional property that is L ortho-Arguesian.

1. INTRODUCTION

The problem of characterizing classical and quantum logics occupies the central place in the development of the theory of quantum structures. These are two approaches to this problem: lattice-theoretic characterizations and probabilistic characterizations. The idea of lattice characterization is due to Birkhoff and von Neumann (1936), who proposed to interpret the lattice of closed subspaces of a Hilbert space as a propositional calculus defining quantum logic. The fundamental theorem on the lattice characterization of a quantum logic is due to Piron (1964) and McLaren (1964). Following them there have been many papers developing this idea; but here we will not discuss the results in this direction.

Another approach to quantum logic is provided by a probabilistic characterization and has been introduced by Mackey (1963), who proposed an axiom system for quantum mechanics expressed in terms of the probability function characterizing the results of measurements for observables and states

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of the considered physical system. The probabilistic approach seems to be more natural from the experimental point of view, since the results of the measurements have statistical character and the information about the system has, in general, a probabilistic nature. Also the approach of Bell (1964) and Clauser *et al.* (1969) expressing the characteristic properties of classical and quantum systems in terms of the inequalities involving the probability of events and their correlation fits into the probabilistic framework. The idea of characterizing classical probability systems by means of inequalities can be found already in a paper of Boole (1862). An algorithm based upon Boole's method for the characterization of classical probabilities has been developed recently by Del Noce (n.d.). A full analysis of the inequalities related to Bell and Clauser–Horne inequalities can be found in a monograph of Pitowsky (1993), where the so-called polytope approach is proposed.

2. CHARACTERIZATION OF CLASSICAL PROBABILITIES

Here we recall a method for characterization of all probability inequalities corresponding to a classical system.

For $N = \{1, \ldots, n\}$ we define the set of Bell functions of order n by

$$B(n) = \{\epsilon: 2^n \to \{-1, 0, 1\} \,|\, \forall_{X\subseteq N} \sum_{T\subseteq X} \epsilon(T) \in \{0, 1\}\} \tag{1}$$

with the convention that $\epsilon(\emptyset) = 0$.

We also define the set of correlation functions of order n by

$$C(n) = \{p \,|\, p: 2^N \to [0, 1] \text{ and } p(\emptyset) = 0\} \tag{2}$$

For each $\epsilon \in B(n)$ and each $p \in C(n)$ let

$$L_\epsilon(p) := \sum_{T\subseteq N} \epsilon(T)p(T) \tag{3}$$

A correlation function $p \in C(n)$ is said to be classically representable if there is a Boolean algebra A, a probability measure m: $A \to [0, 1]$, and a sequence $a_1, a_2, \ldots, a_n$ of elements of A such that for each $T \subseteq N$

$$p(T) = m\left(\bigwedge_{i\in T} a_i\right) \tag{4}$$

We denote the set of all classically representable correlation functions of order n by $c(n)$.

The following theorem has been proved in Beltrametti and Mączyński (1993).

Theorem 1.

$$p \in c(n) \Leftrightarrow \forall \epsilon \in B(n), \quad 0 \le L_\epsilon(p) \le 1 \tag{5}$$

The inequality $0 \le L_\epsilon(p) \le 1$ defined in (5) is called a Bell-type inequality. Hence p is classically representable iff Bell-type inequalities corresponding to p hold for all Bell functions of order n. Theorem 1 can be used to generate and verify all Bell-type inequalities.

In any case we can say that the problem of probabilistic characterization of classical logics (corresponding to Boolean algebras) has been fully solved.

3. NONCLASSICAL PROBABILITIES

A generalization of Bell-type inequalities for some orthomodular lattices has been recently proposed by Länger *et al.* (n.d.), but the problem of probabilistic characterization of orthomodular logics is far from a full solution. In this paper we would like to discuss this problem for the special case of correlation sequences involving only pairwise correlations. Using the polytope approach of Pitowsky (1989), we will show that in this special case no Bell-type inequalities for orthomodular lattices can be found, except the trivial inequalities $0 \le p_i \le 1$ and $0 \le p_i - p_{ij} \le 1$, $0 \le p_j - p_{ij} \le 1$.

Following Pitowsky, instead of the general correlations functions (1), we will consider correlation sequences in order n defined only for pairs (i, j) belonging to some fixed subset S of the set of all pairs:

$$p = (p_1, \ldots, p_n, \ldots, p_{ij}, \ldots) \tag{6}$$

where $(i, j) \in S \subseteq \{(i, j) | i < j,\ i, j = 1, \ldots, n\}$.

The terms of this sequences are real numbers, so p belongs to the $(n + |S|)$-dimensional real vector space $R^{n+|S|}$. We will denote this space by $R(n, S)$.

We say that $p \in R(n, S)$ is classically representable if there is a Boolean algebra A, a probability measure $m: A \to [0, 1]$, and a sequence of elements $a_1, \ldots, a_n \in A$ such that

$$p_i = m(a_i), \qquad i = 1, \ldots, n$$
$$p_{ij} = m(a_i \wedge a_j) \qquad \text{for all } (i, j) \in S$$

The set of all classically representable $p \in R(n, S)$ will be denoted by $c(W, S)$.

Similarly, we say that $p \in R(n, S)$ is quantum representable if there is a Hilbert space H, a trace-class self-adjoint operator W of trace 1, and a sequence of projections $P_1, \ldots, P_n$ in H such that

$$p_i = \mathrm{tr}(W P_i) \qquad \text{for} \quad i = 1, \ldots, n$$
$$p_{ij} = \mathrm{tr}(W(P_i \wedge P_j)) \qquad \text{for} \quad (i, j) \in S$$

We denote the set of all quantum representable $p \in R(n, S)$ by $q(n, S)$.

Finally, $l(n, S)$ denotes the "limiting" set of correlation sequences:

$$l(n, S) := \{p \in R(n, S) \mid 0 \leq p_i \leq 1, 0 \leq p_{ij} \leq \min\{p_i, p_j\}\}$$

Pitowsky (1989) showed that

$$c(n, S) \subseteq q(n, S) \subseteq l(n, S)$$

where $c(n, S)$ is convex and closed, $q(n, S)$ is convex but not closed, and $l(n, S)$ is obviously convex and closed. He also showed that

$$\overline{q(n, S)} = l(n, S)$$

The sets $c(n, S)$, $q(n, S)$, and $l(n, S)$ are called classical, quantum, and limiting polytopes, respectively. The quantum polytope $q(n, S)$ should be more properly called Hilbert space polytope, and it is of course a nonclassical polytope. In this paper we would like to consider the most general nonclassical polytope $g(n, S)$ corresponding to probabilities defined on a generalized probability space. Namely, p belongs to $g(n, S)$ iff there is a generalized probability space (L, m), where L is an orthomodular lattice $L = (L, \wedge, \vee, {}^{\perp}, 0, 1)$ and m is a probability measure $m: L \to [0, 1]$, and there is a sequence of elements $a_1, \ldots, a_n \in L$ such that

$$p_i = m(a_i), \qquad i = 1, \ldots, n$$
$$p_{ij} = m(a_i \wedge a_j) \qquad \text{for} \quad (i, j) \in S$$

For the reason of physical interpretation of L as a space of events we will additionally assume that L admits a full set of probability measures, i.e., there is a set $\{m_\alpha : \alpha \in M\}$ of probability measures on L such that $a \leq b$ in L iff $m_\alpha(a) \leq m_\alpha(b)$ for all $\alpha \in M$. This ensures that the order structure of L can be defined by probabilities.

We have obviously

$$c(n, S) \subseteq q(n, S) \subseteq g(n, S) \subseteq l(n, S)$$

and

$$\overline{g(n, S)} = \overline{q(n, S)} = l(n, S)$$

The main theorem of this paper is as follows

Theorem 2.

$$g(n, S) = l(n, S)$$

Theorem 2 states that there is no limitation on probabilities to be represented in generalized probability spaces based on orthomodular lattices except for the trivial limitations $0 \leq p_i \leq 1$ and $0 \leq p_{ij} \leq \min\{p_i, p_j\}$, contrary to the situation in classical probability theory, where probabilities are restricted by Bell-type inequalities.

An outline of the proof of Theorem 2 goes as follows.

Let $p = (p_1, \ldots, p_n, \ldots, p_{ij}, \ldots) \in l(n, S)$.

We have to find a representation for p in a generalized probability space. We shall consider the cases $n = 2$, $n = 3$, and $n \geq 4$ (for $n = 1$ the theorem trivially holds).

1. The case $n =$ z. We have $p = (p_1, p_2, p_{12})$ with $0 \leq p_i \leq 1$, and $0 \leq p_{12} \leq p_1$, $0 \leq p_{12} \leq p_2$. We consider an orthomodular lattice defined by the Greechie diagram shown in Fig. 1. This lattice consists of 12 elements and is built by gluing two 8-element Boolean algebras $2^{\{a_1, b_1, c\}}$ and $2^{\{a_2, b_2, c\}}$ so that the atom c and coatom $c^\perp$ are common to both (together with 0 and 1 elements). For the details of building orthomodular lattices from blocks of Boolean algebras by Greechie's method see, e.g., Pták and Pulmannová (1992).

Figure 1 defines an orthomodular lattice because there is no loop of order 3 or 4. We define a probability measure m on L_{12} by setting weights on the atoms:

$$\begin{aligned} m(a_i) &= 1 - p_i, \qquad i = 1, 2 \\ m(b_i) &= p_i - p_{12}, \qquad i = 1, 2 \\ m(c) &= p_{12} \end{aligned}$$

By a theorem in Pták and Pulmannová (1992), this mapping extends uniquely to a probability measure on L_{12}. We now take $A_1 = a_1^\perp$, $A_2 = a_2^\perp$. We have

$$\begin{aligned} m(A_i) &= m(a_i^\perp) = m(b_i \vee c) = 1 - m(a_i) \\ &= 1 - (1 - p_i) = p_i, \qquad i = 1, 2 \\ m(A_1 \wedge A_2) &= m(c) = p_{12} \end{aligned}$$

so (L, m) is a representation for p. It is evident that the lattice L_{12} admits a full set of probability measures, even a full set of $\{0, 1\}$-valued probability measures.

2. The case $n = 3$. We have $p = (p_1, p_2, p_3, p_{12}, p_{13}, p_{23})$ with $0 \leq p_i \leq 1$, $0 \leq p_{ij} \leq \min\{p_i, p_j\}$. We define an orthomodular lattice by the

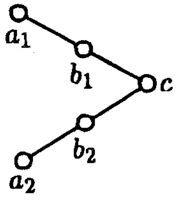

Fig. 1. The Greechie diagram of the lattice L_{12}.

Greechie diagram in Fig. 2. By Greechie's theorem this diagram defines an orthomodular lattice, since there are no loops of order 3 and 4 (there is a loop of order 6). We define a probability measure m on L_{26} as indicated in Fig. 2. We take

$$\begin{aligned} A_1 &= a_1^\perp = b \vee c, & &\text{hence } m(A_1) = 1 - (1 - p_1) = p_1 \\ A_2 &= a_2^\perp = b \vee d, & &\text{hence } m(A_2) = p_2 \\ A_3 &= a_3^\perp = c \vee d, & &\text{hence } m(A_3) = p_3 \end{aligned}$$

We also have

$$\begin{aligned} m(A_1 \wedge A_2) &= m(b) = p_{12} \\ m(A_1 \wedge A_3) &= m(c) = p_{13} \\ m(A_2 \wedge A_3) &= m(d) = p_{23} \end{aligned}$$

Hence (L_{26}, m) is a representation for p. Observe that similarly as L_{12}, the lattice L_{26} admits a full set of $\{0, 1\}$-probability measures.

3. The case $n \geq 4$. For simplicity we shall consider in detail the case $n = 4$; the generalization for higher n is obvious by induction. We have

$$p = (p_1, \ldots, p_n, \ldots, p_{ij}, \ldots) \qquad (ij) \in S$$
$$0 \leq p_i \leq 1, \qquad 0 \leq p_{ij} \leq \min\{p_i, p_j\}$$

Consider a Greechie diagram in the form of a $2n$-gon with $n(n - 3)/2$ diagonals. For $n = 4$ this looks as shown in Fig. 3. This Greechie diagram defines an orthomodular lattice, since there are no loops of order 3 and 4. We may assume that $S = S_0 = \{(i, j) | i < j,\ i, j = 1, \ldots, n\}$, since otherwise S can be extended to S_0 by defining $p_{ij} = 0$ for $(i, j) \in S_0 \backslash S$. The number of elements in this lattice is

$$|L| = 6\binom{n}{2} + 2n + 2, \qquad n \geq 2$$

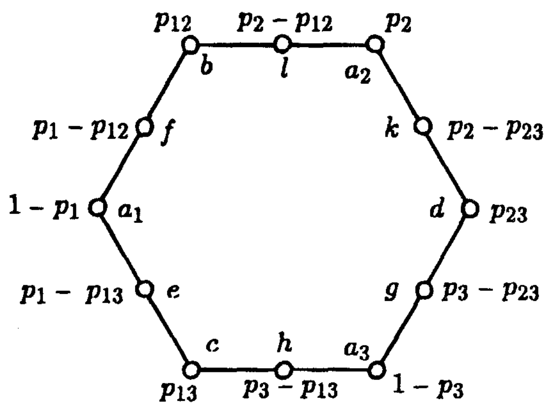

Fig. 2. The Greechie diagram of the lattice L_{26}.

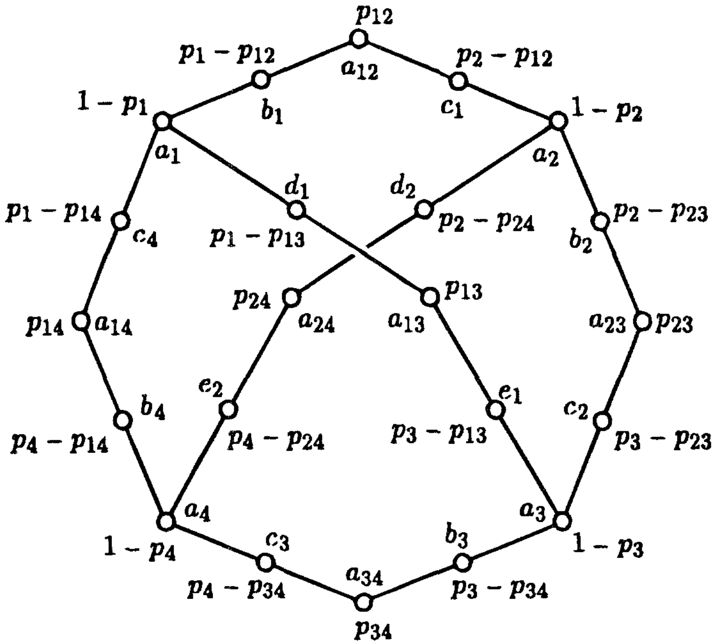

Fig. 3. The Greechie diagram of the lattice L_{46}.

giving for $n = 4$

$$|L| = 46.$$

We denote the lattice of Fig. 3 by L_{46}. We define a probability measure m on L_{46} by assigning weights to atoms as indicated in Fig. 3. We now take

$$A_i = a_i^{\perp}, \qquad i = 1, \ldots, 4$$

We have

$$A_i \wedge A_j = a_i^{\perp} \wedge a_j^{\perp} = (b_i \vee a_{ij}) \wedge (c_i \vee a_{ij}) = a_{ij} \qquad \text{for} \quad 1 \leq i < j \leq 4$$

Hence

$$m(A_i) = m(a_i^{\perp}) = 1 - (1 - p_i) = p_i$$
$$m(A_i \wedge A_j) = m(a_{ij}) = p_{ij}$$

which shows that (L, m) is a representation for p. It is not difficult to show that the lattice L admits a full set of $\{0, 1\}$-valued probability measures. Hence $p \in g(4, S_0)$ and also $p \in g(4, S)$. In this way we have proved Theorem 2 for $n = 4$. The case $n > 4$ is analogous. Hence Theorem 2 holds for $n \geq 2$ and this ends the proof.

Besides admitting a full set of $\{0, 1\}$-probability measures, the lattices used in the proof of Theorem 2 enjoy the additional property of being ortho-Arguesian. Following Alan Day and R. Greechie (see Greechie, 1979), we say that an orthomodular lattice L is ortho-Arguesian if for any three orthogonal pairs $(a_i, b_i) \in L$, $i = 1, 2, 3$, $a_i \perp b_i$, the elements

$$
\begin{aligned}
x &= (a_1 \vee b_1) \wedge (a_3 \vee b_3) \\
c_i &= (a_j \vee a_k) \wedge (b_j \vee b_k), \\
y &= c_3 \wedge (c_1 \vee c_2) \qquad \text{where } \{i, j, k\} = \{1, 2, 3\} \\
z &= (a_1 \wedge (a_2 \vee y)) \vee b_1
\end{aligned}
$$

fulfill $x \leq z$. Greechie (1979) has given an example of an orthomodular lattice which admits a full set of $\{0, 1\}$-probability measures but is not ortho-Arguesian, hence it cannot be embedded in the lattice $L(H)$ of closed subspaces of a Hilbert space [the lattice $L(H)$ is ortho-Arguesian]. It is not difficult to show that all the lattices in the proof of Theorem 2 are ortho-Arguesian [for details, see Greechie (1979)]. In this sense they are quite regular, but it is an open question whether they are embeddable in the lattice $L(H)$, i.e., whether they are standard logics or not. We understand the embedding in the weak sense, because the embedding in the strong sense—together with probability measures—is not possible. This is due to the fact that the probability measures on $L(H)$ have the Jauch–Piron property: if E, F are projections and $m(E) = m(F) = 1$, then $m(E \wedge F) = 1$; whereas probability measures used to represent $p \in g(n, S)$ may not have this property. We can even have $m(a) = m(b) = 1$ with $m(a \wedge b) = 0$. This also shows that the difference between $g(n, S)$ and $q(n, S)$ lies in a boundary set of probability measures without the Jauch–Piron property. For a discussion of this problem see Beltrametti and Mączyński (n.d.). We conclude with the remark that Theorem 2 shows that the framework of arbitrary orthomodular lattices is so general that no restrictions—except the trivial ones—are imposed on the values of probabilities, in contradistinction to the probabilities on Boolean algebras and Hilbert spaces. This applies to probabilities involving correlations for pairs only, leaving the problem of representing triple and higher correlations open.

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